JOURNAL OF APPROXIMATION THEORY 80, 272–297 (1995)

Generalized Refinement Equations and Subdivision Processes

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Communicated by Allan Pinkus

Received August 21, 1992; accepted in revised form December 19, 1993

The concept of subdivision schemes is generalized to schemes with a continuous mask, generating compactly supported solutions of corresponding functional equations in integral form. A necessary and a sufficient condition for uniform convergence of these schemes are derived. The equivalence of weak convergence of subdivision schemes with the existence of weak compactly supported solutions to the corresponding functional equations is shown for both the discrete and integral cases. For certain non-negative masks stronger results are derived by probabilistic methods. The solution of integral functional equations whose continuous masks solve discrete functional equations, are shown to be limits of discrete nonstationary schemes with masks of increasing support. Interesting functions created by these schemes are C^∞ functions of compact support including the *up*-function of Rvachev. © 1995 Academic Press, Inc.

1. INTRODUCTION

Functional equations (FE) of the form,

$$f(x) = \sum_{j \in \mathbb{Z}} a_j f(2x - j), \tag{1.1}$$

* Research supported by Israel Ministry of Science and Absorption.

0021-9045/95 $6.00

are closely related to subdivision processes (SP) of the form $f^{k+1} = Sf^k$, where

$$f_j^{k+1} = \sum_{i \in \mathbb{Z}} a_{j-2i} f_i^k, \qquad j \in \mathbb{Z}. \tag{1.2}$$

We say that the SP (1.2) is convergent if for the initial data $f_i^0 = \delta_{0,i}$ there exists a function $\phi \in C(\mathbb{R})$, $\phi \not\equiv 0$, such that

$$\lim_{k \to \infty} \sup_{j \in \mathbb{Z}} |f_j^k - \phi(2^{-k} j)| = 0.$$

The function ϕ is termed "the basic function of the SP," and for any general initial data f^0, the SP converges to $\sum_{j \in \mathbb{Z}} f_j^0 \phi(\cdot - j)$.

If the SP (1.2) is convergent, and $\{a_j\}$ is a compactly supported mask, then the above function ϕ is, up to normalization, the unique compactly supported solution of the FE (1.1). Also, $\operatorname{supp}\{\phi\}$ is contained in the convex hull of $\operatorname{supp}\{a_j\}$. Thus the SP is an efficient way of generating the compactly supported solution of the FE. The analysis of the FE, and the properties of its solutions can also be done via the SP, [CDM; DGL2; DaL1, 2; DL1].

A necessary condition for the convergence of the SP, in the above sense, is that

$$\sum_{j \in \mathbb{Z}} a_{2j} = \sum_{j \in \mathbb{Z}} a_{2j+1} = 1. \tag{1.3}$$

This, however, is not a necessary condition for the existence of a compactly supported, or even smooth, solution of the FE. Derfel [D1] investigated the solution of more general FE of the form

$$f(x) = \sum_j a_j f(\alpha_j x - \beta_j), \tag{1.4}$$

and even further generalized FE, namely

$$f(x) = \int_{-\infty < \lambda < \infty} \int_{0 < \mu < \infty} f\left(\frac{x - \lambda}{\mu}\right) F(d\lambda, d\mu), \tag{1.5}$$

where $F(d\lambda, d\mu)$ is a probability measure on $\mathbb{R}_+^2$. Derfel [D1] uses probability tools, and gives criteria for the existence of non-trivial bounded continuous solutions of this equation, and in particular for the FE (1.4) with $a_j \geqslant 0$ and $\sum a_j = 1$.

In this work we first consider a special case of (1.5) which is an integral generalization of (1.1), where the weights $\{a_j\}$ are replaced by a weight function a, namely the integral functional equation (IFE)

$$f(x)=\int_{-\infty}^{\infty} a(t)\, f(2x-t)\, dt=(a*f)(2x). \tag{1.6}$$

In parallel we introduce a corresponding integral subdivision process (ISP) generalizing (1.2), of the form

$$f^{k+1}(x)=\int_{-\infty}^{\infty} a(x-2t)\, f^k(t)\, dt=(a(2\cdot)*f^k)\left(\frac{x}{2}\right). \tag{1.7}$$

While in the discrete SP (1.2) we use f^k to denote a vector in $l^\infty(\mathbb{Z})$, in the ISP (1.7) f^k denotes a function. The process starts with an initial function $f^0(x)$, and generates a sequence of functions $\{f^k(x)\}$. We say that the ISP is convergent if the sequence $\{h^k(x)\}$, defined by

$$h^k(x)=f^k(2^k x),$$

is uniformly convergent. The relation between the IFE and the ISP is much more transparent than in the discrete case. We show that if

$$\Pi(\lambda)=\lim_{k\to\infty}\prod_{j=1}^{k}\left(\tfrac{1}{2}\,\hat{a}(\lambda 2^{-j})\right) \tag{1.8}$$

exists and is in L^1, where $\hat{a}$ is the Fourier transform of a, then there exists a solution f of (1.6), which is the inverse Fourier transform of $\Pi(\lambda)$,

$$f(x)=\Pi(\lambda)^\vee.$$

Furthermore, the convergence in L^1 of the infinite product (1.8) is a sufficient condition for the convergence of the ISP, and if we start with $f^0(x)=\delta(x)$ we get

$$h^k(x)=f^k(2^k x)=\left(\prod_{j=1}^{k}\tfrac{1}{2}\,\hat{a}(\lambda 2^{-j})\right)^\vee,$$

and hence

$$\lim_{k\to\infty} h^k(x)=\lim_{k\to\infty} f^k(2^k x)=f(x). \tag{1.9}$$

To unify the treatment of the discrete and integral FE and SP we introduce Stieltjes IFE and ISP.

Let A and F be functions of bounded variation, and let $\#$ denote the Stieltjes convolution. Consider the Stieltjes integral functional equation (SFE)

$$F(x) = \int_{-\infty}^{\infty} F(2x - t)\, dA(t) = (F \# A)(2x), \tag{1.10}$$

where $\int_{-\infty}^{\infty} dA(t) = 1$. In parallel we introduce a corresponding Stieltjes integral subdivision process (SSP) of the form

$$F^{k+1}(x) = \int_{-\infty}^{\infty} A(x - 2t)\, dF^k(t) = (A(2\cdot) \# F^k)\left(\frac{x}{2}\right), \tag{1.11}$$

and consider the convergence of the sequence $\{H^k(x)\}$, defined by

$$H^k(x) = F^k(2^k x). \tag{1.12}$$

The analysis of the SSP–SFE involves the infinite Stieltjes convolution

$$F = A(2\cdot) \# A(4\cdot) \# \cdots \# A(2^k\cdot) \# \cdots. \tag{1.13}$$

We show that if this infinite convolution converges pointwise, or in a weak sense, then the limit is a solution of the SFE in the same sense. The functions generated by the SSP with $F^0(x) = \Theta(x) = \frac{1}{2}(1 + \operatorname{sgn}(x))$ are of the form

$$H^k = A(2\cdot) \# A(4\cdot) \# \cdots \# A(2^k\cdot), \qquad k = 1, 2, ...$$

and their limit exists and solves the SFE, in the appropriate sense.

The relation between SSP–SFE and ISP–IFE is clear if we weaken the notion of convergence of the ISP to convergence in the distributional sense (weak convergence). We show that if the SSP is convergent in the distributional sense, and $F = \lim_{k \to \infty} H^k$, with $F^0(x) = \Theta(x)$, then the distributional derivative of F, say f, is the solution, in the distributional sense, of the IFE with $a = 2A'$ as the weight function. Furthermore, the sequence generated by the corresponding ISP satisfies

$$h^k(x) = \frac{d}{dx} H^k(x), \qquad k = 1, 2, ...,$$

where all the derivatives above are distributional derivatives.

The atomic case of SSP with

$$A(x) = \tfrac{1}{2} \sum_m a_m \Theta(x - m),$$

induces the notion of weak convergence of the SP (1.2). Thus, if the SSP, with $F^0 = \Theta$, converges weakly to F, then the distributional derivative $f = F'$ solves weakly the FE (1.1) with the mask $\{a_m\}$. Also, the SP (1.2) converges weakly to f, that is, if f^k is generated by the SP (1.2) with the mask $\{a_m\}$, starting with $f_j^0 = \delta_{j,0}$, $j \in \mathbb{Z}$, then

$$\lim_{k \to \infty} 2^{-k} \sum_{i \in \mathbb{Z}} f_i^k g(2^{-k} i) = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx,$$

for all test functions $g \in C_0^\infty$.

The above observations lead to the conclusion that the existence of a weak compactly supported solution of the FE (IFE) is equivalent to the weak convergence of the corresponding SP (ISP).

Following the result in [DaL1] we show that,

$$\sum_{i \in \mathbb{Z}} a_i = 2^n, \qquad n \in \mathbb{Z}_+,$$

is necessary and sufficient for the existence of a weak compactly supported solution of the FE, and conclude that this condition with $n = 1$ is also necessary and sufficient for the weak convergence of the SP. Similar conclusions hold for the IFE-ISP. In particular if the discrete SP (1.2), with the mask $\{a_m\}$, is convergent, then F' is defined as a function and is the limit of the SP for the initial data $f_j^0 = \delta_{j,0}$.

Thus by allowing a weaker sense of convergence we obtain a full analogy between solutions of the functional equations and the convergence of the corresponding subdivision processes.

Probabilistic tools can be efficiently used for the investigation of the convergence of the infinite convolution (1.13). For each A a non-degenerate probability distribution function, with a finite first absolute moment, it is shown that the SSP is always uniformly convergent to the unique continuous solution of the SFE in the class of probability distributions. If, moreover, A has a density $\frac{1}{2}a$, then the limit function is in C^∞ and its derivative is the limit of the ISP with weight function a. There is a transparent probabilistic interpretation of SSP in that case. Let us define the random geometrical series

$$z = \sum_{n=1}^{\infty} 2^{-n} \xi_n,$$

and let us denote by z_k the partial sums of order k. Here ξ_n are independent identically distributed random variables with probability distribution A. Then $F = \lim_{k \to \infty} H^k$ is the probability distribution function of z, and H^k are the probability distribution functions of z_k. Other applications of probabilistic methods to subdivision processes were developed in [B1].

The ISP (1.7) is not a practical way for computing the solution of the IFE (1.6). Surprisingly, it can be shown that if the weight function a is itself a solution of a discrete FE of the form (1.1), dilated by a factor 2, then there is a discrete non-stationary SP which converges (at least weakly) to the solution of the IFE. This yields a method for computing solutions of IFE such as the Rvachev *up*-function and its generalizations [R1, 2].

We conclude the introduction by two model examples related to the main results of this paper.

EXAMPLE — THE *up*-FUNCTION. Consider the IFE (1.6) with the weight function

$$a(x) = \chi_{[-1,1]}(x) = \begin{cases} 1, & -1 \leqslant x \leqslant 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.14}$$

with the Fourier transform

$$\hat{a}(\lambda) = 2\,\frac{\sin \lambda}{\lambda}.$$

The IFE (1.6) takes the form

$$f(x) = \int_{-1}^{1} f(2x - t)\, dt = \int_{2x-1}^{2x+1} f(t)\, dt, \tag{1.15}$$

and by differentiation

$$f'(x) = 2[f(2x+1) - f(2x-1)]. \tag{1.16}$$

Thus the integral functional equation (1.15) is equivalent to the differential functional equation (1.16). The solution of these equations is Rvachev's *up*-function, $up(x) \in C_0^\infty$ [R1], with the Fourier transform

$$\hat{up}(\lambda) = \prod_{j=1}^{\infty} \frac{\sin(\lambda 2^{-j})}{\lambda 2^{-j}}. \tag{1.17}$$

Note that $a(x) = B_0((x+1)/2)$, where B_0 is the B-spline of order 1, is obtained as the limit of a SP with mask $a_m = \delta_{0,m} + \delta_{1,m}$. Therefore, as we show later, the *up*-function can be obtained as the limit of a non-stationary discrete SP. At level k the mask of this non-stationary SP is the mask of the stationary SP generating the B-spline of order k (see Theorem 18). In [R1] Rvachev iterates (1.15) to obtain a sequence of function which converges uniformly to the *up*-function. Starting with $\sigma_0(x)=\frac{1}{2}a(x)$ he defines for $n\geqslant 0$

$$\sigma_{n+1}(x)=\int_{-\infty}^{\infty} a(t)\,\sigma_n(2x-t)\,dt=\int_{2x-1}^{2x+1}\sigma_n(t)\,dt,$$

and proves that $\sigma_n(x)\to up(x)$ uniformly. Here $\operatorname{supp}\{\sigma_n\}=(-1,1)$, and $\int_{-1}^{1}\sigma_n(x)\,dx=1$. The ISP with the mask (1.14) generates another sequence of functions which also converges uniformly to the *up*-function (see Theorem 2). Starting with $h^0(x)=\delta(x)$ the ISP generates the sequence $\{h^k\}$ by the rule

$$h^{k+1}(x)=\int_{-\infty}^{\infty} a(2t)\,h^k(x-2^{-k}t)\,dt=\int_{-1/2}^{1/2} h^k(x-2^{-k}t)\,dt$$

$$=2^k\int_{x-2^{-k-1}}^{x+2^{-k-1}} h^k(t)\,dt.$$

Note that $h^1(x)=a(2x)$, $\operatorname{supp}\{h^k\}=(-1+2^{-k},1-2^{-k})$, $k\geqslant 1$, and that $\int_{-1}^{1}h^k(x)\,dx=1$.

In general one may consider iterating the functional equation operator,

$$\sigma_{n+1}(x)=(a*\sigma_n)(2x),\qquad n\in\mathbb{Z}_+,$$

starting with some σ_0, as a mean for generating the solution of the IFE. It can be easily verified that these iterations with $\sigma_0=\delta$, generate a sequence $\{\sigma_n\}$ which is identical with $\{h^n\}$ generated by the ISP with $h^0=\delta$.

EXAMPLE — WEAK CONVERGENCE. Consider the mask

$$a_m=\delta_{1,m}+\delta_{-1,m}, \tag{1.18}$$

which does not satisfy the necessary condition for convergence (1.3). The FE (1.1) takes the form

$$f(x)=f(2x+1)+f(2x-1), \tag{1.19}$$

and it has the solution

$$f(x)=\begin{cases}\frac{1}{2}, & -1\leqslant x\leqslant 1,\\ 0, & \text{otherwise.}\end{cases}$$

The SP (1.2) with the mask (1.18), starting with $f_j^0=\delta_{0,j}$, produces the values

$$f_j^k=\begin{cases}1, & j \text{ odd } -2^k\leqslant j\leqslant 2^k,\\ 0, & j \text{ even},\end{cases}\qquad k>0.$$

These values do not converge pointwise to f. Yet, we can use these values to form the functional

$$\langle h^k, g\rangle = 2^{-k} \sum_i f_i^k g(2^{-k} i), \qquad g \in C_0^\infty .$$

Then it is easy to see that

$$\lim_{k \to \infty} \langle h^k, g\rangle = \tfrac{1}{2} \int_{-1}^{1} g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

for all $g \in C_0^\infty$, verifying the convergence in the distributional sense of the SP to the solution of (1.19).

2. The Solution of IFE and the Convergence of ISP

The correspondence between the IFE and the ISP is made clear in the Fourier domain. Let $a \in L^1(-\infty, \infty)$, and let $f \in L^1(-\infty, \infty)$ be the solution of the IFE (1.6),

$$f(x) = (a * f)(2x). \tag{2.1}$$

The Fourier transform of f satisfies the functional equation

$$\hat{f}(\lambda) = \frac{1}{2} \hat{a}\left(\frac{\lambda}{2}\right) \hat{f}\left(\frac{\lambda}{2}\right). \tag{2.2}$$

Iterating on (2.2) we find out that formally

$$\hat{f}(\lambda) = \lim_{k \to \infty} \hat{f}(\lambda 2^{-k}) \prod_{j=1}^{k} \left(\tfrac{1}{2} \hat{a}(\lambda 2^{-j})\right). \tag{2.3}$$

Hence, if we have convergence in (2.3) then

$$\hat{f}(\lambda) = C \prod_{j=1}^{\infty} \left(\tfrac{1}{2} \hat{a}(\lambda 2^{-j})\right) \tag{2.4}$$

with $C = \hat{f}(0) = \int_{-\infty}^{\infty} f(x)\, dx$.

On the other hand consider the ISP (1.7) which we can write as

$$f^{k+1}(2x) = \int_{-\infty}^{\infty} a(2x - 2t)\, f^k(t)\, dt = (a(2\cdot) * f^k)(x). \tag{2.5}$$

By applying the Fourier transform here we have

$$\hat{f}^{k+1}\left(\frac{\lambda}{2}\right)=\hat{a}\left(\frac{\lambda}{2}\right)\hat{f}^{k}(\lambda). \tag{2.6}$$

Now, for $h^k(x)=f^k(2^k x)$, (2.6) becomes

$$\hat{h}^{k+1}(\lambda)=\tfrac{1}{2}\,\hat{a}(2^{-k-1}\lambda)\,\hat{h}^k(\lambda).$$

Hence, starting with $h^0(x)=f^0(x)=C\,\delta(x)$, $\hat{h}^0(\lambda)=C$, we obtain

$$\hat{h}^k(\lambda)=C\prod_{j=1}^{k}\left(\tfrac{1}{2}\,\hat{a}(\lambda 2^{-j})\right), \qquad k\in\mathbb{Z}_+. \tag{2.7}$$

We thus derive the following Theorem:

THEOREM 1. *If the infinite product* $\prod_{j=1}^{\infty}(\frac{1}{2}\hat{a}(\lambda 2^{-j}))$ *is convergent in* $L^1(-\infty,\infty)$ *to* $\hat{f}$, *and if we apply the ISP* (1.7) *starting with* $f^0(x)=\delta(x)$, *then*

$$\lim_{k\to\infty} f^k(2^k x)=f(x),$$

uniformly on $\mathbb{R}$, *where* f *is a solution of the IFE* (1.6).

Remark — The Condition $\hat{a}(0)=2$.
By (2.3) and (2.7) we observe that $\hat{a}(0)=2$ is a necessary condition for convergence (in L^1) of the ISP, and for the existence of a solution $f\in L^1$ to the IFE with $\hat{f}(0)\neq 0$.

THEOREM 2. *If a has a finite support* $[-L, L]$, *then*:

1. *A necessary condition for uniform convergence of the ISP, to a nontrivial limit, is* $\int a(t)\,dt=2$.

2. *The ISP converges uniformly to a* C_0^∞ *function if* $\int a(t)\,dt=2$, $\int |a(t)|\,dt<4$ *and* $|\hat{a}(\lambda)|=O(|\lambda|^{-r})$, *as* $|\lambda|\to\infty$, *for some* $r>0$.

Proof. The first result is easily derived by using the relation

$$h^{k+1}(x)=\int_{-\infty}^{\infty} h^k(x-2^{-k}t)\,a(2t)\,dt,$$

which follows from (2.5) To prove the second result we first show by induction that

$$|h^k(x)-h^k(y)|\leqslant \alpha^{k-1}M\,|x-y|,$$

for $k > k_0$, where $\alpha = \int |a(2t)|\, dt = \frac{1}{2}\int |a(t)|\, dt$ and M is a positive constant. Starting the ISP with $h^0(x) = \delta(x)$, we get by (2.7) that $|h^k(\lambda)| = O(\lambda^{-kr})$ as $\lambda \to \infty$. Hence, for some k_0, $h^{k_0} \in C^1(\mathbb{R})$, and the induction hypothesis holds for $k = k_0$. For $k > k_0$ we use the relation

$$|h^{k+1}(x) - h^{k+1}(y)| = \left|\int_{-\infty}^{\infty} a(2t)(h^k(x - 2^{-k}t) - h^k(y - 2^{-k}t))\, dt\right|$$
$$\leqslant \alpha \cdot \alpha^{k-1} M\, |x - y|.$$

Hence,

$$|h^{k+1}(x) - h^k(x)| = \left|\int_{-\infty}^{\infty} a(2t)(h^k(x - 2^{-k}t) - h^k(x))\, dt\right|$$
$$\leqslant \int |a(2t)|\, dt \cdot \max_{2t \in [-L, L]} |h^k(x - 2^{-k}t) - h^k(x)|$$
$$\leqslant \alpha^k M 2^{-k} \frac{L}{2}.$$

It thus follows that $\{h^k\}$ is uniformly convergent for $\alpha < 2$. Repeating the above arguments it can be shown that for any $m \in \mathbb{Z}_+$ the sequence of derivatives $\{(d^m/dx^m)\, h^k(x)\}$, for $k \geqslant k_0(m)$, exists and is uniformly convergent. Thus the limit is in $C_0^{\infty}(\mathbb{R})$. ∎

Remark — The Decay Conditions on $\hat{a}$. As follows from the next theorem, much weaker conditions on $\hat{a}$, namely the two conditions

$$|\hat{a}(\lambda) - 2| = 0(|\lambda|^{\gamma}), \qquad \text{as} \quad |\lambda| \to 0, \qquad \gamma > 0, \tag{2.8}$$

and

$$|\hat{a}(\lambda)| = o(1) \qquad \text{as} \quad |\lambda| \to \infty, \tag{2.9}$$

are sufficient for the solution of the IFE to be in C^{∞}. However, the stronger decay conditions on $\hat{a}$ in Theorem 2 for the existence of $k_0(m) \in \mathbb{Z}_+$ such that $\hat{h}^k \in C^m(\mathbb{R})$ for $k > k_0(m)$, cannot in general be relaxed. It is easy to verify that this condition is necessary if $|\hat{a}(\lambda)|$ is monotonically decreasing on $\mathbb{R}_+$.

THEOREM 3. *Suppose* $\hat{a}$ *satisfy* (2.8) *and* $|\hat{a}(\lambda)| < q \leqslant 2^{-n-\varepsilon}$ *for* $|\lambda| \geqslant \lambda_0$, $\varepsilon > 0$. *If* $f \in L^1$ *is a solution of the IFE* (1.6) *then* $f \in C^n(\mathbb{R})$.

Proof. $\hat{f}$ can be represented by the infinite product (2.4), which by condition (2.8) converges pointwise to a uniformly bounded function on compact sets. It follows that $\hat{f}$ satisfies the functional equation (2.2).

Let $C = \sup_{|\lambda| < \lambda_0} |\hat{f}(\lambda)|$ and let $2^m \lambda_0 < |\lambda| < 2^{m+1} \lambda_0$, then by (2.2)

$$\begin{aligned} |\hat{f}(\lambda)| &= 2^{-m} \left| \hat{a}\left(\frac{\lambda}{2}\right) \hat{a}\left(\frac{\lambda}{4}\right) \cdots \hat{a}\left(\frac{\lambda}{2^m}\right) \hat{f}\left(\frac{\lambda}{2^{m+1}}\right) \right| \\ &\leqslant \left(\frac{q}{2}\right)^m C \leqslant \left(\frac{q}{2}\right)^{\log_2 |\lambda|} C' \\ &= |\lambda|^{\log_2 q - 1} C' \leqslant C' \, |\lambda|^{-n-1-\varepsilon}, \end{aligned}$$

proving that $\lambda^n \hat{f}(\lambda) \in L^1$ and hence that $f \in C^n(\mathbb{R})$. ∎

Obviously if $\hat{a}$ satisfies (2.9), then $f \in C^n(\mathbb{R})$ for any $n \in \mathbb{Z}_+$.

We now consider the correspondence between the SFE (1.10) and the SSP (1.11) starting with $F^0(x) = \Theta(x)$, when the notions of solutions and convergence are weakened.

Let K be the space of C_0^∞ functions with the topology as in [GS], and let D denote the space of distributions on K. For $F \in D$ we denote by $\langle F, g \rangle$ the operation of F on $g \in C_0^\infty$. If $F \in L^1_{loc}$ then

$$\langle F, g \rangle = \int_{-\infty}^{\infty} g(x) \, F(x) \, dx.$$

Consider the set of locally supported distributions:

$$D_0 = \{ F \in D \mid \exists \Omega_F \text{ bounded, s.t. } \langle F, g \rangle = 0 \text{ if } \operatorname{supp}\{g\} \cap \Omega_F = \varnothing \}.$$

For $F, G \in D_0$, we define $F \# G = F * G'$, i.e.,

$$\langle F \# G, h \rangle = \langle F, g \rangle,$$

where $h \in C_0^\infty$ and

$$g(x) = \langle G', h(x + \cdot) \rangle \in C_0^\infty.$$

The following properties are used below:

$$\text{(i)} \qquad F \# G = G \# F,$$

$$\text{(ii)} \quad \langle F(a \cdot), g \rangle = a^{-1} \langle F, g(a^{-1} \cdot) \rangle,$$

$$\text{(iii)} \quad (G \# F)(a \cdot) = G(a \cdot) \# F(a \cdot).$$

We say that the SSP is weakly convergent if, for any initial F^0, there exists $F \in D$ s.t.

$$\lim_{k \to \infty} \langle H^k, g \rangle = \langle F, g \rangle, \qquad \forall g \in C_0^\infty.$$

Similarly we say that $F \in D$ is a weak solution of the SFE (1.10) if

$$\langle F, g \rangle = \langle F \# A(2\cdot), g \rangle, \tag{2.10}$$

for any test function $g \in C_0^\infty$.

We are particularly interested in the class of normalized compactly supported distributions:

$$D_0^1 = \{F \mid F \in D,\ F' \text{ is of compact support } \Omega_F,\ \langle F', g \rangle = 1 \text{ if } g|_{\Omega_F} = 1\}.$$

Clearly, if $F, G \in D_0^1$ then $F \# G \in D_0^1$.

THEOREM 4. *Consider a SFE and a SSP with $A \in D_0^1$. Then the following three statements are equivalent:*

A. *There exists a weak solution $F \in D_0^1$ of the SFE,*

B. *The SSP, starting with $F^0(x) = \Theta(x)$, is weakly convergent to the distribution $F \in D_0^1$, defined by*

$$\langle F, g \rangle = \lim_{k \to \infty} \langle H^k, g \rangle, \qquad \forall g \in C_0^\infty.$$

C. *The infinite convolution*

$$A(2\cdot) \# A(4\cdot) \# \cdots \# A(2^n\cdot) \# \cdots \tag{2.11}$$

is weakly convergent to $F \in D_0^1$.

Proof. Using (1.11) and (1.12) we have

$$H^k(x) = (A(2^k\cdot) \# A(2^{k-1}\cdot) \# \cdots \# A(2\cdot) \# \Theta)(x). \tag{2.12}$$

Hence, if C hold, then $\lim_{k \to \infty} H^k(\cdot) = F(\cdot)$, where the convergence is in the weak sense, and B also holds. F is also a weak solution of the SFE since

$$\begin{aligned}\langle (F \# A)(2\cdot), g \rangle &= \langle F(2\cdot) \# A(2\cdot), g \rangle \\ &= \langle A(2\cdot) \# F(2\cdot), g \rangle \\ &= \langle A(2\cdot) \# (A(4\cdot) \# A(8\cdot) \# \cdots \# A(2^{n+1}\cdot) \# \cdots), g \rangle \\ &= \langle F(x), g \rangle.\end{aligned}$$

Hence A follows. If we assume that A holds, then iterating (2.10) we obtain

$$\langle F, g\rangle = \langle A(2\cdot) \# (A(4\cdot) \# A(8\cdot) \# \cdots \# A(2^n\cdot) \# F(2^n\cdot), g\rangle.$$

Now, since $\langle 2^n F'(2^n\cdot), g\rangle = \langle F', g(2^{-n}\cdot)\rangle$, and $F \in D_0^1$, we have $\langle 2^n F'(2^n\cdot), g\rangle \to g(0)$, $\forall g \in C_0^\infty$, that is $F(2^n\cdot) \to \Theta$ as $n \to \infty$, and C follows. The proof is completed by observing that each $H^k \in D_0^1$, hence C follows if B holds. ∎

In the discrete (atomic) case,

$$A(x) = \tfrac{1}{2} \sum_m a_m \Theta(x-m), \tag{2.13}$$

the SSP generates the functions

$$H^k(x) = 2^{-k} \sum_i f_i^k \Theta(x - 2^{-k} i),$$

where f^k is generated by the SP with the mask $a = \{a_m\}$. Hence, the weak convergence of the SSP implies that for any test function $g \in C_0^\infty$, and for $h^k = (H^k)'$,

$$\lim_{k\to\infty} \langle h^k, g\rangle = \lim_{k\to\infty} 2^{-k} \sum_i f_i^k\, g(2^{-k} i) = \langle f, g\rangle,$$

thus inducing the notion of weak convergence of SP.

Remark 5. Consider the SFE

$$F(x) = (F \# A)(2x).$$

If $F \in D_0$ solves this equation with weight $A \in D_0^1$, then $F^{(m)}$ solves the equation $F^{(m)}(x) = (F^{(m)} \# 2^m A)(2x)$, $m \in \mathbb{N}$. Furthermore, if the corresponding subdivision process, starting with $F^0 = \Theta(x)$, converges to F, i.e.,

$$H^k(x) = (A(2^k\cdot) \# A(2^{k-1}\cdot) \# \cdots \# A(2\cdot) \# \Theta)(x) \to F(x),$$

then the same process, starting with $F^0(x) = \delta^{(m-1)}(x)$, converges to $F^{(m)}$.

COROLLARY 6. *Necessary and sufficient conditions for the existence of a compactly supported weak solution f of the FE (1.1) is that*

$$\sum_j a_j = 2^n, \qquad n \in \mathbb{N}. \tag{2.14}$$

Furthermore, for $A(x)$ *defined by* (2.13) *the SSP with normalized weight*

$$\tilde{A}(x) = 2^{-n+1} A(x) \in D_0^1$$

starting with $F^0 = \delta^{(n-2)}$, *converges weakly to* f (f *has at most* $(n-1)$*th order singularity*).

Proof. Sufficiency of (2.14) follows directly from the result in [DaL1] for the case $n = 1$, and Remark 5. To prove necessity, let us suppose that y is a compactly supported weak solution of (1.1). Then the generalized Fourier transform $\hat{y}(\lambda)$ is an entire function of order 1, finite type, with polynomial growth on the real line. Moreover, $\hat{y}$ is a solution of the functional equation

$$\hat{y}(\lambda) = p\left(\frac{\lambda}{2}\right) \hat{y}\left(\frac{\lambda}{2}\right),$$

where

$$p(\lambda) = \tfrac{1}{2} \sum_j a_j e^{ij\lambda}.$$

Then, in the neighborhood of the origin (see for example [PS])

$$\hat{y}(\lambda) = \lambda^{\ln N/\ln 2 - 1} Q(\lambda),$$

where $Q(\lambda)$ is a function analytic at the origin, and $N = \sum_j a_j$. Hence $\hat{y}(\lambda)$ is analytic only if $\ln N/\ln 2 - 1 \in \mathbb{Z}_+$, i.e., $N = \sum_j a_j = 2^n$, where $n \in \mathbb{N}$. The proof of the second statement of the corollary follows from Remark 5. ∎

More general functional equations of the form

$$\sum_{j=0}^{l} \sum_{k=0}^{m} a_{j,k} y^{(k)}(\alpha_j x + \beta_j) = 0, \tag{2.15}$$

are considered in [D2]. A necessary condition for the existence of compactly supported solutions of (2.15) is

$$\sum_{j=0}^{l} a_{j,0} \alpha_j^{-n} = 1, \qquad \text{for some} \quad n \in \mathbb{N}. \tag{2.16}$$

Remark 7. The result of Corollary 6 carries through to the case of IFE and ISP if the weight function a is of compact support, and

$$\int_{-\infty}^{\infty} a(x)\, dx = 2^n, \qquad n \in \mathbb{N}.$$

3. Non-negative Masks — Probabilistic methods

Let P denote the class of probability distribution functions on $\mathbb{R}$. If $A \in P$ then Theorem 4 can be strengthen as follows:

Theorem 8. *Consider a SFE and a SSP with $A \in P$. Then the following statements are equivalent:*

A. *There exists a solution $F \in P$ of the SFE.*

B. *The SSP, starting with $F^0 = \Theta(x)$, generates a sequence of functions $\{H^k\}_{k \geq 0} \subset P$ converging pointwise to $F \in P$ at all points of continuity of F.*

C. *The infinite convolution* (2.11) *converges to $F \in P$ at all points of continuity of F.*

Moreover, if $A \in P$ satisfies a certain moment condition the convergence of the infinite convolution (2.11) is ensured, and we can also say something about the continuity and uniqueness of the solution of the SFE. To do this we use probability tools, in particular the following Theorems by Zakusilo [Z1, 2], on the limit distributions of some random geometric series.

Consider the random geometrical series

$$y = \sum_{n=1}^{\infty} t^{-n} \eta_n, \tag{3.1}$$

where $\{\eta_n\}$ are independent, identically distributed, random variables with probability distribution A.

Theorem 9. [Z1] *If*

$$M(|\eta|) = \int_{-\infty}^{\infty} |x| \, dA(x) < \infty, \tag{3.2}$$

then the random series (3.1) *converges with probability* 1.

In the next two theorems condition (3.2) is assumed to hold.

Theorem 10. [Z2] *If $y \neq c$, then the probability distribution function of y is continuous, and it is of pure type. i.e., it is purely absolutely continuous, or it is purely singular continuous.*

By Lebesque Theorem any $A \in P$ can be decomposed into three components of pure type

$$A = p_1 A_a + p_2 A_{sc} + p_3 A_{ac}, \tag{3.3}$$

where A_a is atomic, A_{sc} is singular continuous, and A_{ac} is absolutely continuous, with $p_i \geqslant 0$ and $\sum p_i = 1$. In the following we use the decomposition

$$A = qA_s + pA_{ac}, \qquad q \geqslant 0, \qquad p \geqslant 0, \qquad p + q = 1, \tag{3.4}$$

where $A_s = (p_1 A_a + p_2 A_{sc})/(1 - p_3)$, $p = p_3$.

EXAMPLE — CANTOR'S FUNCTION. Consider the random series

$$y = \sum_{n=1}^{\infty} 3^{-n} \eta_n,$$

where $\{n_\eta\}$ are independent, identically distributed, random variables with probability distribution A defined by $P\{\eta_n = 0\} = P\{\eta_n = 2\} = \frac{1}{2}$. Then the probability distribution function of y, $F_y(x)$, is the famous Cantor's singular continuous function [F]. This function is continuous, monotonically increasing on $[0, 1]$, with a derivative vanishing almost everywhere. It satisfies the SFE [D1]

$$F_y(x) = \int_{-\infty}^{\infty} F_y(3x - t)\, dA(t) = (F_y \# A)(3x) = \tfrac{1}{2} F_y(3x) + \tfrac{1}{2} F_y(3x - 2).$$

This follows from an analogue of Theorem 8 for general t in (3.1).

THEOREM 11 [Z2]. *If $A = qA_s + pA_{ac}$, and $q < t^{n+1}$, then the probability distribution function of y, $F(x)$ has a density $f(x)$ which is n times differentiable (continuous when $n = 0$).*

Using the above results and Theorem 8 we obtain,

THEOREM 12. *If A is a non-degenerate probability distribution function of a random variable ξ, such that $M(|\xi|) < \infty$, then the SSP converges uniformly to a continuous function $F \in P$. Moreover, F is of pure type, and is the unique solution of the SFE (1.10) in P.*

Proof. Let us define the random geometrical series

$$z = \sum_{n=1}^{\infty} 2^{-n} \xi_n, \tag{3.5}$$

where $\{\xi_n\}$ are independent, identically distributed, random variables with probability distribution A. Let z_k denote the partial sum of order k of z. By (2.12) H^k is the probability distribution function of the random variable z_k. By Theorems 8 and 9 $\lim_{k \to \infty} H^k$ exists pointwise and is a continuous function F, which is the probability distribution of the random variable z. In fact, due to the continuity of F, the convergence of $\{H^k\}$ to F is uniform by Polya's Theorem ([B, Remark 3, p. 142]).

By Theorem 8, F is a solution of the SFE. To prove its uniqueness in P, let us assume that $G(x)$ is a probability distribution function solving (1.10). Iterating with the SFE we get

$$G(x) = (A(2\cdot) \# A(4\cdot) \# \cdots \# A(2^n\cdot) \# G(2^n\cdot))(x).$$

We now observe that for any probability distribution function G, $\lim_{n\to\infty} G(2^n x) = \Theta(x)$, and consequently the r.h.s. of the above expression for $G(x)$ tends to $F(x)$. ∎

Remark 13. By [G], all the above conclusions still hold if the condition $M(|\xi|) < \infty$ is replaced by

$$M(\ln \sup(\xi, 1)) = \int_{-\infty}^{\infty} \ln \sup(x, 1)\, dA(x) < \infty. \tag{3.6}$$

COROLLARY 14. *Let A be the probability distribution function of an atomic random variable ξ, i.e., $A(x)$ is a step function with jumps $\{a_j\}$ at the points $\{x_j\}$, $a_j \geqslant 0$, $\sum a_j = 1$. Then the SSP converges uniformly to a continuous function F, which is the unique solution in P of the functional equation*

$$F(x) = \sum_j a_j F(2x - x_j). \tag{3.7}$$

As a direct consequence of Theorem 11, we get

COROLLARY 15. *If A has a density function a of compact support, then the sequence of functions $h^k(x) = f^k(2^k x)$, obtained by the ISP*

$$f^{k+1}(x) = 2\int_{-\infty}^{\infty} a(x - 2t)\, f^k(t)\, dt, \tag{3.8}$$

starting with $f^0(x) = \delta(x)$, converges in the distributional sense to an infinitely differentiable probability density function, which is the unique solution of the IFE

$$f(x) = 2\int_{-\infty}^{\infty} a(t)\, f(2x - t)\, dt = 2(a * f)(2x). \tag{3.9}$$

in the class of probability density functions.

Note that by Theorem 2, uniform convergence of $\{h^k\}$ is guaranteed if $\hat{a}(\lambda) = O(|\lambda|^{-r})$ as $|\lambda| \to \infty$, for some $r > 0$.

4. Solution of IFE as the Limit of a Non-stationary SP

We now investigate the possibility of finding a discrete SP which also converges to the solution of the IFE. The discrete SP we derive is however non-stationary, i.e., it is of the form

$$f_j^{k+1} = \sum_j \beta_{j-2i}^k f_i^k, \qquad j \in \mathbb{Z}, \tag{4.1}$$

with weights depending on the subdivision level k.

Let us define the trigonometric polynomials

$$\beta^k(\lambda) = \sum_n \beta_m^k e^{-im\lambda}, \qquad k \geqslant 0, \tag{4.2}$$

and apply the process to the initial data is $f_j^0 = \delta_{0,j}$. Let us also assume that $\beta^k(0) = 2$, $k \geqslant 0$. Then, as shown in [DL3], if the process is weakly convergent, then its basic function f has the Fourier transform

$$\hat{f}(\lambda) = \prod_{k=0}^{\infty} \left(\tfrac{1}{2}\beta^k(\lambda 2^{-k-1})\right). \tag{4.3}$$

On the other hand we have the formal infinite product representation (2.4) for the Fourier transform of an L^1 solution of the IFE (1.6). We would like to identify weight functions a for which the infinite product (2.4) can be rewritten in the form (4.3), where each β^k is a trigonometric polynomial of a finite degree. In the following we present a formal derivation of the correspondence between an IFE and a non-stationary SP. Later we give more concrete results for special classes of weight functions.

Theorem 16. *Let $a(2\cdot) \in L^1(\mathbb{R})$ be the basic function of a non-stationary SP with coefficients $\{\alpha_m^k\}$ satisfying*

$$\sum_j \alpha_j^k = 2. \tag{4.4}$$

Let $\alpha^k(\lambda) = \sum_m \alpha_m^k e^{-im\lambda}$, and define another non-stationary SP with coefficients $\{\beta_m^k\}$ determined by the trigonometric equality

$$\beta^k(\lambda) = \sum_m \beta_m^k e^{-im\lambda} = 2 \prod_{l=0}^{k} \left(\tfrac{1}{2}\alpha^l(\lambda)\right). \tag{4.5}$$

If the last non-stationary SP converges weakly to an L^1 limit, then the limit function, with the initial data $f_j^0 = \delta_{0,j}$ is the solution of the IFE (1.6), with $a(x)$ as a weight function.

Proof. By (4.3)

$$\hat{a}\left(\frac{\lambda}{2}\right)=2\prod_{k=0}^{\infty}\left(\frac{1}{2}\alpha^k(\lambda 2^{-k-1})\right). \tag{4.6}$$

To show that f is a solution of the IFE we show that it satisfies (2.2). We use (4.3), (4.5), and (4.6):

$$\begin{aligned}\frac{1}{2}\hat{a}\left(\frac{\lambda}{2}\right)\hat{f}\left(\frac{\lambda}{2}\right)&=\prod_{k=0}^{\infty}\left(\frac{1}{2}\alpha^k(\lambda 2^{-k-1})\right)\prod_{k=0}^{\infty}\left(\frac{1}{2}\beta^k(\lambda 2^{-k-2})\right)\\&=\prod_{k=0}^{\infty}\left(\frac{1}{2}\alpha^k(\lambda 2^{-k-1})\right)\prod_{k=0}^{\infty}\prod_{l=0}^{k}\left(\frac{1}{2}\alpha^l(\lambda 2^{-k-2})\right)\\&=\prod_{k=0}^{\infty}\prod_{l=0}^{k}\left(\frac{1}{2}\alpha^l(\lambda 2^{-k-1})\right)=\hat{f}(\lambda).\end{aligned}$$ ∎

EXAMPLE — THE *up*-FUNCTION AS THE LIMIT OF A NON-STATIONARY SP. Let us recall the example of the *up*-function, which solves the IFE with the weight function a given by (1.14). The function $a(2\cdot-1)$ is a solution of the FE (1.19), and is also the basic function of a SP with the mask $a_m=\delta_{0,m}+\delta_{1,m}$. Then the solution of the IFE with $a(\cdot-1)$ is $up(\cdot-1)$. The corresponding trigonometric polynomials α^k are simply $\alpha^k(\lambda)=1+e^{-i\lambda}$, and the subsequent polynomials β^k given by (4.5) are

$$\beta^k(\lambda)=2^{-k}(1+e^{-i\lambda})^{k+1}.$$

Therefore, the *up*-function can be obtained as the weak limit of a non-stationary SP (4.1) with coefficients

$$\beta_m^k=2^{-k}\binom{k+1}{m},\qquad 0\leqslant m\leqslant k.$$

In fact it can be shown that this limit is strong, namely this non-stationary SP converges uniformly (see Theorem 18 below).

The *up*-function of Rvachev is a C^∞ function on $\mathbb{R}$, with a support $[-1,1]$, and, as shown above, it is the limit of a non-stationary SP with masks whose supports grow linearly. For fixed k the mask β^k constitutes the mask of a stationary SP with the B-spline of degree k as its basic function. By (4.5) it is clear that the support of the mask $\{\beta_m^k\}$ is always growing.

The next result extends the case of the *up*-function, and it gives a probabilistic interpretation to the relationship between the solution in P of an SFE and the limit of a non-stationary SSP.

THEOREM 17. *Let $G(2\cdot)\in P$ be the solution of a SFE (1.10), where $A\in P$ is non-degenerate, and has finite first absolute moment, and let H be the solution of a SFE with G as the weight function. Then, H can be obtained as the limit of a non-stationary SSP*

$$F^{k+1}(x)=\int_{-\infty}^{\infty} A_k(x-2t)\,dF^k(t), \qquad F^0=\Theta, \tag{4.7}$$

with

$$A_k=A_{k-1}\#A=\underbrace{A\#A\#\cdots\#A}_{k+1-\text{times}}, \qquad k\geqslant 0, \qquad A_0=A, \tag{4.8}$$

namely

$$\lim_{k\to\infty} F^k(2^k x)=H(x).$$

Furthermore, if G is absolutely continuous then $H\in C^\infty(\mathbb{R})$. In particular $H\in C^\infty(\mathbb{R})$ if A has the form (3.4) with $q<\frac{1}{2}$.

Proof. First we have to show that the non-stationary SSP converges. Similar to the proof of Theorem 4 we derive the relation

$$\begin{aligned} H^k(x)=F^k(2^k x)&=(A_{k-1}(2^k\cdot)\#F^{k-1}(2^{k-1}\cdot))(x)\\ &=(A_{k-1}(2^k\cdot)\#H^{k-1})(x), \end{aligned}$$

and consequently for $F^0=\Theta$.

$$H^k(x)=(A_{k-1}(2^k\cdot)\#A_{k-2}(2^{k-1}\cdot)\#\cdots\#A_0(2\cdot)\#\Theta)(x). \tag{4.9}$$

Let us define the random geometrical series

$$z=\sum_{n=1}^{\infty} 2^{-n}\xi_n, \tag{4.10}$$

where ξ_n is a random variable with probability distribution A_{n-1}. Let z_k denote the partial sum of order k of z, $k\geqslant 1$. By (4.9) H^k is the probability distribution function of the random variable z_k. By an extension of Theorem 9 ([L, Corollary 3.7.3]), the random series (4.10) converges with probability 1 to a proper random variable z. Thus $H=\lim_{k\to\infty} H^k$ exists and is the probability distribution function of z.

To show that H solves the SFE, with the weight G, we should show that

$$H=(G\#H)(2\cdot). \tag{4.11}$$

Consider also the random geometrical series

$$y = \sum_{n=1}^{\infty} 2^{-n} \eta_n, \tag{4.12}$$

where $\{\eta_n\}$ are independent, identically distributed, random variables with probability distribution A. Then, by Theorem 4, $G(2\cdot)$ is the probability distribution function of y, and by Theorem 12 G is continuous. Since $H(2x)$ is the probability distribution of $\frac{1}{2}z$, then $(G \# H)(2\cdot)$ is the probability distribution of $y + \frac{1}{2}z$, which we rewrite as

$$w = y + \tfrac{1}{2}z = 2^{-1}\eta_1 + \sum_{n=2}^{\infty} 2^{-n}(\eta_n + \xi_{n-1}).$$

Now we observe that η_1 and ξ_1 are identically distributed, and so are ξ_n and $\eta_n + \xi_{n-1}$ for $n > 1$. Therefore, w and z are identically distributed and (4.11) follows.

By Theorem 4

$$G(2\cdot) = \cdots \# A(2^k \cdot) \# A(2^{k-1} \cdot) \# \cdots \# A(2\cdot),$$

and

$$H = \cdots \# (G(2^k \cdot) \# G(2^{k-1} \cdot) \# \cdots \# G(2\cdot).$$

Hence Theorem 10 implies that $G \in C(\mathbb{R})$ and therefore also $H \in C(\mathbb{R})$. Furthermore, by Theorem 11, $H \in C^{\infty}(\mathbb{R})$ if G is absolutely continuous. In particular (again by Theorem 11) G is absolutely continuous if A has the form (3.4) with $q < \frac{1}{2}$. ∎

The following result is in part a corollary of Theorem 17 for a special atomic case. It generalizes the example of the *up*-function, where the convergence of the non-stationary SP is in the strong sense.

THEOREM 18. *Let $a(2\cdot)$ be the solution of a FE with a mask $\{\alpha_m\}$ satisfying*

$$\alpha(z) = \sum_{j \in \mathbb{Z}} \alpha_j z^j = (1+z)\, q(z),$$

with $\alpha(1) = 2$, and $q(z)$ a polynomial with non-negative coefficients. Let a non-stationary SP with coefficients $\{\beta_m^k\}$ be defined by

$$\beta^k(z) = \sum_{j \in \mathbb{Z}} \beta_j^k z^k = 2[\alpha(z)/2]^k.$$

Then the scheme is convergent (in the strong sense), and its basic limit function is in C_0^∞. Moreover, this function is a solution of the IFE (1.6) with $a(x)$ as above.

Proof. The second claim follows directly from Theorem 17. To prove the strong convergence to C^∞ functions we use here the machinery of [DGL2], (see also [Dy]). It is sufficient to show that for any $m \in \mathbb{Z}_+$, the scheme determined by the polynomials

$$\beta_m^k(z) = \left(\frac{2}{1+z}\right)^m \beta^k(z), \qquad k \geqslant m,$$

is uniformly convergent. This would imply that the SP $\{\beta^k\}_{k \geqslant 0}$ converges uniformly to C^∞ limit functions. The uniform convergence of $\{\beta_m^k\}_{k \geqslant m}$ is equivalent to the contractivity of $\{q_m^k\}_{k \geqslant m+1}$ where q_m^k is the polynomial

$$q_m^k(z) = \frac{1}{1+z}\,\beta_m^k(z) = 2^{m-k+1}(1+z)^{k-m-1}\,(q(z))^k,$$
$$k \geqslant m+1. \tag{4.13}$$

A sufficient condition for this contractivity is that the coefficient of $q_m^k(z) = \sum_i q_{m,i}^k z^i$ satisfy for all $k > m+1$

$$\sum_i |q_{m,2i}^k| < 1, \qquad \sum_i |q_{m,2i+1}^k| < 1. \tag{4.14}$$

Now, since $\{q_m^k\}$ are polynomials with non-negative coefficients, and since $q_m^k(1) = 1$, it is enough to show that $\sum_i q_{m,2i}^k > 0$ and $\sum_i q_{m,2i+1}^k > 0$. This property is guaranteed for $k > m+1$ by the existence of the factor $(1+z)$ in the right hand side of (4.13). ∎

5. Non-stationary ISP and Systems of IFE

Non-stationary binary SP of the form

$$f_j^{k+1} = \sum_{i \in \mathbb{Z}} \alpha_{j-2i}^k f_i^k, \qquad j \in \mathbb{Z}, \tag{5.1}$$

has been considered in [DL2, 3]. While the limit of a stationary SP is related to the solution of a FE, the limit of the process (5.1) is related to the solution of a system of functional equations

$$\phi_m(x) = \sum_i \alpha_i^m \phi_{m+1}(2x - i), \qquad m = 0, 1, 2, \ldots. \tag{5.2}$$

The function ϕ_m is the basic function of the non-stationary SP $\{\alpha^{k+m}\}_{k \geq 0}$.
Here we consider more general non-stationary SSP of the form

$$\begin{aligned} F^{k+1}(x) &= \int_{-\infty}^{\infty} A_k\left(x - \frac{p_{k+1}}{p_k} t\right) dF_k(t) \\ &= \left[A_k\left(\frac{p_{k+1}}{p_k} \cdot\right) \# F^k(\cdot)\right]\left(\frac{p_k}{p_{k+1}} x\right) \\ & \qquad k = 0, 1, 2, \ldots, \end{aligned} \tag{5.3}$$

where A_k, F^k are functions of bounded variation. That is, at each level we use another weight function, A_k, and also another dilation factor p_{k+1}/p_k, where $p_0 = 1$, $p_k > 1$ for $k > 1$. Starting with $\Theta(x)$, we analyze the convergence of the sequence of functions $\{H^k\}$,

$$H^k(x) = F^k(p_k x). \tag{5.4}$$

In parallel we consider the infinite system of SFE

$$\begin{aligned} U_k(x) &= \int_{-\infty}^{\infty} U_{k+1}\left(\frac{p_{k+1}}{p_k} x - t\right) dA_k(t) \\ &= [U_{k+1} \# A_k]\left(\frac{p_{k+1}}{p_k} x\right), \qquad k = 0, 1, 2, \ldots. \end{aligned} \tag{5.5}$$

The key to the convergence of the SSP, and to the existence of a solution to the system (5.5), is the convergence of the infinite convolution

$$A_0(p_1 \cdot) \# A_1(p_2 \cdot) \# \cdots \# A_n(p_{n+1} \cdot) \# \cdots. \tag{5.6}$$

THEOREM 19. *If the infinite convolution* (5.6) *converges to F in some sense - uniformly, pointwise, weakly - then*:

A. *The SSP* (5.3)–(5.4) *converges, in the same sense, to F,*

$$\lim_{k \to \infty} H^k = F.$$

B. *A solution of the system* (5.5) *is given by*

$$U_k(x) = [A_k(p_{k+1}\cdot) \# A_{k+1}(p_{k+2}\cdot) \# \cdots \\ \# A_{k+n}(p_{k+n+1}\cdot) \# \cdots]\left(\frac{x}{p_k}\right), \qquad k \geq 0. \tag{5.7}$$

Proof. From (5.3)–(5.4) it follows that

$$H^{k+1}(x) = F^{k+1}(p_{k+1}x) = \left[A_k\left(\frac{p_{k+1}}{p_k}\cdot\right) \# F^k(\cdot)\right](p_k x) \\ = [A_k(p_{k+1}\cdot) \# F^k(p_k\cdot)](x) = [A_k(p_{k+1}\cdot) \# H^k](x).$$

Therefore,

$$H^{k+1}(x) = [A_k(p_{k+1}\cdot) \# A_{k-1}(p_k\cdot) \# \cdots \# A_0(p_1\cdot) \# \Theta](x),$$

and thus A follows. To prove B we rewrite the system (5.5) as

$$U_k(x) = [A_k \# U_{k+1}]\left(\frac{p_{k+1}}{p_k}x\right), \qquad k = 0, 1, 2, ...,$$

and it is clear that $\{U_k\}_{k \geq 0}$ given by (5.7) is a solution. ∎

The convergence of the infinite convolution (5.6) can be ensured when A_k are probability distribution functions.

THEOREM 20. *Let $A_k(x)$ be a probability distribution function of a random variable η_k, such that*

$$M_k = \int_{-\infty}^{\infty} |x|\, dA_k(x) < \infty, \qquad k \geq 0,$$

and let the numbers $\{p_k\}_{k \geq 0}$ satisfy $\sum_{k=0}^{\infty} M_k p_{k+1}^{-1} < \infty$. Then the infinite convolution (5.6), *and hence the SSP* (5.3)–(5.4) *converges to $F \in P$ pointwise at every point of continuity of F.*

Proof. Consider the random series

$$z_k = p_k \sum_{n=k}^{\infty} \frac{\eta_n}{p_{n+1}}, \tag{5.8}$$

where $\{\eta_n\}$ are independent random variables, with probability distributions $\{A_n\}$ respectively. Each z_k is a proper random variable since $\sum_{k=0}^{\infty} M_k p_{k+1}^{-1} < \infty$ (by Corollary 3.7.3 in [L]). Thus the infinite convolution (5.7) converges to the probability distribution function of z_k. Also, the function H^m, generated in the mth step of the SSP (5.3)–(5.4), is the probability distribution of the partial sum of order m of z_0. In particular $F = U_0 = H^\infty$ is the probability distribution of z_0.

Remark 21. *The solution to the system* (5.5) *is not unique even in P. It is easy to see that if* $\{U_k\}$ *with* $U_k \in P$ *is a solution of* (5.5), *then for any* $C \in P$, $\{U_k \# C(\cdot/p_k)\}_{k \geq 0}$ *is also a solution of* (5.5), *and* $U_k \# C(\cdot/p_k) \in P$.

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Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis